Rainbow connection in some digraphs

Jesús Alva-Samos[1] Juan José Montellano-Ballesteros[2]

Abstract

An edge-coloured graph G is *rainbow connected* if any two vertices are connected by a path whose edges have distinct colours. This concept was introduced by Chartrand et al. in [3], and it was extended to oriented graphs by Dorbec et al. in [5]. In this paper we present some results regarding this extention, mostly for the case of circulant digraphs.

Keywords: arc-coloring; rainbow connected; connectivity

1 Introduction

Given a connected graph $G = (V(G), E(G))$, an edge-coloring of G is called *rainbow connected* if for every pair of distinct vertices u, v of G there is a uv-path all whose edges received different colors. The *rainbow connectivity number of* G is the minimum number $rc(G)$ such that there is a rainbow connected edge-coloring of G with $rc(G)$ colors. Similarily, an edge-coloring of G is called *strong rainbow connected* if for every pair $u, v \in V(G)$ there is a uv-path of minimal length (a uv-geodesic) all whose edges received different colors. The *strong rainbow connectivity number of* G is the minimum number $src(G)$ such that there is a strong rainbow connected edge-coloring of G with $src(G)$ colors.

The concepts of rainbow connectivity and strong rainbow connectivity of a graph were introduced by Chartrand et al. in [3] and, been the connectivity one fundamental notion in Graph Theory, it is not surprising that several works around these concepts has been done since then (see for instance [2, 4, 6, 7, 8, 9, 10, 11, 12]). For a survey in this topic see ([13]). As a natural extension of this notions is that of

[1] Instituto de Matemáticas, UNAM
[2] Instituto de Matemáticas, UNAM. juancho@matem.unam.mx

the *rainbow connection* and *strong rainbow connection* in oriented graphs, which was introduced by Dorbec et al. in [5].

Let $D = (V(D), A(D))$ be a strong connected digraph and $\Gamma : A(D) \to \{1, \ldots, k\}$ be an arc-coloring of D. Given $x, y \in V(D)$, a directed xy-path T in D will be called *rainbow* if no two arcs of T receive the same color. Γ will be called *rainbow connected* if for every pair of vertices $x, y \in V(D)$ there is a rainbow xy-path and a rainbow yx-path. The *rainbow connection number of* D, denoted as $rc^*(D)$, is the minimum number k such that there is a rainbow connected arc-coloring of D with k colors. Given a pair of vertices $x, y \in V(D)$, an xy-path T will be called an *xy-geodesic* if the length of T is the distance, $d_D(x, y)$, from x to y in D. An arc-coloring of D will be called *strongly rainbow connected* if for every pair of distinct vertices x, y of D there is a rainbow xy-geodesic and a rainbow yx-geodesic. The *strong rainbow connection number of* D, denoted as $src^*(D)$, is the minimum number k such that there is a strong rainbow connected arc-coloring of D with k colors.

In this paper we present some results regarding this problem, mainly for the case of circulant digraphs. For general concepts we may refer the reader to [1].

2 Some remarks and basic results on biorientations of graphs

Let $D = (V(D), A(D))$ be a strong connected digraph of order n and let $\operatorname{diam}(D)$ be the diameter of D. As we see in [5], it follows that

$$\operatorname{diam}(D) \leq rc^*(D) \leq src^*(D) \leq n.$$

Also, it is not hard to see that if H is a strong spanning subdigraph of D, then $rc^*(D) \leq rc^*(H)$. However, as in the graph case (see[2]), this is not true for the strong rainbow connection number, as we see in the next lemma.

Lemma 2.1. *There is a digraph D and a spanning subdigraph H of D such that $src^*(D) > src^*(H)$.*

Proof. Let H be as in Figure 1, where D is obtained from H by adding the arc a_1a_2. It is not hard to sse that the colouring in Figure 1 is a strong rainbow 6-coloring of H, thus $src^*(H) \leq 6$. We will show that $src^*(D) \geq 7$. Suppose there is a strong rainbow 6-coloring ρ of D, First notice that, for each i and j, the u_iv_j-geodesic is unique and contains the arcs u_iv_i and u_jv_j, hence there are no two arcs of the type u_iv_i sharing the same colour. Without loss of generality let $\rho(u_iv_i) = i$ for $1 \leq i \leq 4$. By an analogous argument, since $P_i = u_iv_ia_1a_2u_4v_4$ is the only u_iv_4-geodesic for $i \leq 3$, and $a_1a_2, a_2u_4 \in A(P_i)$, we can suppose that such arcs have colours 5 and 6, respectively. If we assign any of the six colours to the arc v_1a_1, we see that for some $j \geq 2$ the unique u_1v_j-geodesic is no rainbow, contradicting the choice of ρ. Hence $src^*(G) \geq 7$ and the result follows. $\square$

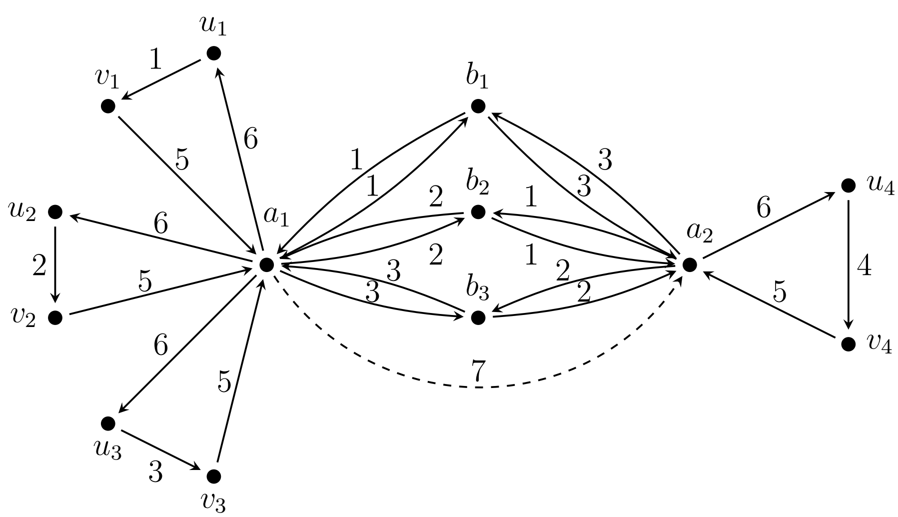

Figure 1: The digraphs D and H from Lemma 2.1.

Given a pair $v, u \in V(D)$, if the arcs uv and vu are in D, then we say that uv and vu are *symmetric* arcs. When every arc of D is symmetric, D is called a *symmetric* digraph. Given a graph $G = (V(G), E(G))$, its *biorientation* is the symmetric digraph $\overleftrightarrow{G}$ obtained from G by replacing each edge uv of G by the pair of symmetric arcs uv and vu.

Given a graph G and a (strong) rainbow connected edge-coloring of G, it is not hard to see that the arc-coloring of $\overleftrightarrow{G}$, obtained by assign the color of the edge uv to both arcs uv and vu is a (strong) rainbow connected arc-coloring of $\overleftrightarrow{G}$. Thus $rc^*(\overleftrightarrow{G}) \leq rc(G)$ and $src^*(\overleftrightarrow{G}) \leq src(G)$. Although for some graphs and its biorientations these values coincide (for instance, as we will see, for $n \geq 4$, $rc(C_n) = src(C_n) = rc^*(\overleftrightarrow{C_n}) = src^*(\overleftrightarrow{C_n})$), for other graphs and its biorientations the difference between those values is unbounded, as we see in the case of the stars, where for each $n \geq 2$, $rc(K_{1,n}) = n$ (for each path between terminal vertices we need two colors) and $rc^*(\overleftrightarrow{K_{1,n}}) = src^*(\overleftrightarrow{K_{1,n}}) = 2$ (the colouring that assigns color 1 to the in-arcs of the "central" vertex and assigns color 2 to the ex-arcs of the central vertex is a strong rainbow coloring).

Theorem 2.2. *Let D be a nontrivial digraph, then*

(a) $src^*(D) = 1$ if and only if $rc^*(D) = 1$ if and only if, for some $n \geq 2$, $D = \overleftrightarrow{K_n}$;

(b) $rc^*(D) = 2$ if and only if $src^*(D) = 2$.

Proof. First observe that since D is nontrivial, $rc^*(D) \geq 1$ and therefore if $src^*(D) = 1$ then $rc^*(D) = 1$. If $rc^*(D) = 1$ then $\operatorname{diam}(D) = 1$ and hence $D = \overleftrightarrow{K_n}$ for some $n \geq 2$. On the other hand, if $D = \overleftrightarrow{K_n}$ it follows that every 1-colouring of D is a strong rainbow colouring. Thus $1 \geq src^*(D) \geq rc^*(D) \geq 1$ and (a) follows. For (b), if $src^*(D) = 2$, by (a), $rc^*(D) > 1$ and hence $rc^*(D) = 2$. If $rc^*(D) = 2$, D has a 2-rainbow colouring and, by (a) $D \neq \overleftrightarrow{K_n}$. Therefore for every pair $u, v \in V(D)$, with $d(u, v) \geq 2$, exists a uv-rainbow path of lenght 2, which is also geodesic. Hence $src^*(D) = 2$ and (b) folows. $\square$

Theorem 2.3. *(a) For $n \geq 2$, $rc^*(\overleftrightarrow{P_n}) = src^*(\overleftrightarrow{P_n}) = n - 1$;*

(b) For $n \geq 4$, $rc^*(\overleftrightarrow{C_n}) = src^*(\overleftrightarrow{C_n}) = \lceil n/2 \rceil$

(c) Let $k \geq 2$, if $\overleftrightarrow{K}_{n_1,n_2,\ldots,n_k}$ is the complete k-partite digraph where $n_i \geq 2$ for some i, then $rc^*(\overleftrightarrow{K}_{n_1,n_2,\ldots,n_k}) = src^*(\overleftrightarrow{K}_{n_1,n_2,\ldots,n_k}) = 2$.

Proof. In [3] it is shown that for every $n \geq 4$, $src(C_n) = \lceil \frac{n}{2} \rceil$ and for every $n \geq 2$, $src(P_n) = n-1$. Since $\text{diam}(\overleftrightarrow{P_n}) = n-1$ it follows that $n-1 \leq rc^*(\overleftrightarrow{P_n}) \leq src^*(\overleftrightarrow{P_n}) \leq src(P_n) = n-1$ and the first part of the theorem follows. In an analogous way, if n is even, $\lceil \frac{n}{2} \rceil = \text{diam}(\overleftrightarrow{C_n}) \leq rc^*(\overleftrightarrow{C_n})$ and since $src^*(\overleftrightarrow{C_n}) \leq src(C_n) = \lceil \frac{n}{2} \rceil$, $rc^*(\overleftrightarrow{C_n}) = src^*(\overleftrightarrow{C_n}) = \lceil \frac{n}{2} \rceil$. Let $n = 2k+1$ with $k \geq 2$ and let us suppose there is a rainbow k-colouring ρ of $\overleftrightarrow{C_n}$. Observe that for every $0 \leq i \leq n-1$, $(v_i, v_{i+1}, \ldots v_{i+k})$ is the only $v_i v_{i+k}$-path of length $d(v_i, v_{i+k}) = k$ in $\overleftrightarrow{C_n}$ and therefore the k colours of ρ occurs in each of such geodesic paths. Thus $\rho(v_i v_{i+1}) = \rho(v_{i+k} v_{i+k+1})$ for each $0 \leq i \leq n-1$, which, since $(k, n = 2k+1) = 1$ implies that all the arcs $v_i v_{i+1}$ in $\overleftrightarrow{C_n}$ receive the same color which is a contradiction. Thus $rc^*(\overleftrightarrow{C_n}) \geq k+1 = \lceil \frac{n}{2} \rceil$ and (b) follows. For (c), since $n_i \geq 2$ for some i, then $\overleftrightarrow{K}_{n_1,n_2,\ldots,n_k}$ is not a complete digraph, hence $rc^*(\overleftrightarrow{K}_{n_1,n_2,\ldots,n_k}) \geq 2$. Let $V_1, V_2, \ldots, V_k$ be the k-partition on independent sets of $V(\overleftrightarrow{K}_{n_1,n_2,\ldots,n_k})$, and for each arc uv, with $u \in V_i$ and $v \in V_j$, assign color 1 to uv if $i < j$ and color 2 if $i > j$. Since $\text{diam}(\overleftrightarrow{K}_{n_1,n_2,\ldots,n_k}) = 2$, it is not hard to see that this is a strong rainbow connected 2-coloring and therefore $src^*(\overleftrightarrow{K}_{n_1,n_2,\ldots,n_k}) \leq 2$. □

Theorem 2.4. *Let D be a spanning strong connected subdigraph of $\overleftrightarrow{C_n}$ with $k \geq 1$ asymmetric arcs. Thus*

$$rc^*(D) = \begin{cases} n-1 & \text{if } k \leq 2; \\ n & \text{if } k \geq 3. \end{cases}$$

Moreover, if $k \geq 3$, $rc^*(D) = src^*(D) = n$.

Proof. Let $V(\overleftrightarrow{C_n}) = \{v_0, \ldots, v_{n-1}\}$ and suppose $v_0 v_{n-1} \notin A(D)$. Since D is strong connected the $v_0 v_{n-1}$-path $T = (v_0, v_1, \ldots, v_{n-1})$ is contained in D, thus $\text{diam}(D) \geq n-1$. Therefore, $n-1 \leq rc^*(D) \leq n$. If $k = 1$ we see that $\overleftrightarrow{P_n}$ is a spanning subdigraph of D, hence $n-1 \leq rc^*(D) \leq rc^*(\overleftrightarrow{P_n})$, which by Theorem 2.3 (a) implies that $rc^*(D) = n-1$. Let $k \geq 2$. If $v_{n-1} v_0 \notin A(D)$, since D is strong connected it follows that D is isomorphic to $\overleftrightarrow{P_n}$ which have no asymmetric arcs and thus this is not possible. Therefore $v_{n-1} v_0 \in A(D)$. If there is a $(n-1)$-rainbow coloring ρ of D, since $v_{n-1} v_0 \in A(D)$, the directed cycle C induced by $A(T) \cup v_{n-1} v_0$ is a

spanning subdigraph of D and therefore there are two arcs $v_iv_{i+1}, v_jv_{j+1} \in A(C)$ such that $\rho(v_iv_{i+1}) = \rho(v_jv_{j+1})$. Since ρ is a rainbow coloring, there is a rainbow v_iv_{j+1}-path and a rainbow v_jv_{i+1}-path in D. Thus the paths $(v_i, v_{i-1}, \dots, v_{j+2}, v_{j+1})$ and $(v_j, v_{j-1}, \dots, v_{i+2}, v_{i+1})$ most be contained in D and therefore the number of assymetric arcs in D is at most 2. Thus, if $k \geq 3$ then $rc^*(D) \geq n$ and hence, $rc^*(D) = n$. Finally, if $k = 2$, let ρ be the $(n-1)$-arc coloring of D which assigns the same color to the assymetric arcs, and for the remaining $n-2$ pairs of simmetric arcs and the remaining $n-2$ colors, ρ assigns the same color to each pair of simmetric arcs. It is not hard to see that ρ is a rainbow coloring of D, thus $rc^*(D) \leq n-1$ and the first part of the theorem follows. The second part is directly from the first part of the theorem and from the fact that $src^*(D) \leq n$. □

As a direct corollary of the previous result we have

Corollary 2.5. *Let D be a strong connected digraph with $m \geq 3$ arcs. Thus $rc^*(D) = src^*(D) = m$ if and only if $D = \vec{C}_m$.*

3 Circulant digraphs

For an integer $n \geq 2$ and a set $S \subseteq \{1, 2, \dots, n-1\}$, the *circulant digraph* $C_n(S)$ is defined as follows: $V(C_n(S)) = \{v_0, v_1, \dots, v_{n-1}\}$ and

$$A(C_n(S)) = \{v_iv_j : j - i \overset{n}{\equiv} s,\ s \in S\},$$

where $a \overset{n}{\equiv} b$ means: *a congruent with b modulo n*. The elements of S are called *generators*, and an arrow v_iv_j, where $j - i \overset{n}{\equiv} s \in S$, will be called an *s-jump*. If $s \in S$ we denote by $C_{(s)}$ the spanning subdigraph of $C_n(S)$ induced by all the s-jumps. Observe that for every pair of vertices v_i and v_j there is at most one v_iv_j-path in $C_{(s)}$. If such v_iv_j-path in $C_{(s)}$ exists will be denoted by $v_iC_{(s)}v_j$. From now on the subscripts of the vertices are taken modulo n. Given an integer $k \geq 1$, let $[k] = \{1, 2, \dots, k\}$.

Theorem 3.1. *If $1 \leq k \leq n-2$, then $rc^*(C_n([k])) = src^*(C_n([k])) = \lceil \frac{n}{k} \rceil$.*

Proof. Let $D = C_n[k]$. The case when $k = 1$ is proved in Theorem 2.4. Let $2 \leq k \leq n-2$, and $V(D) = \{v_0, \ldots, v_{n-1}\}$. By definition it follows that for every pair $0 \leq i \leq j \leq n-1$, $d(v_i, v_j) = d(v_0, v_{j-i})$ and $d(v_j, v_i) = d(v_0, v_{i+n-j})$. Also it is not hard to see that for every $0 \leq i \leq n-1$, $d(v_0, v_i) = \lceil \frac{i}{k} \rceil$. From here it follows that $\mathrm{diam}(D) = \lceil \frac{n-1}{k} \rceil$.

Let $P = \{V_1, V_2, \ldots, V_{\lceil \frac{n}{k} \rceil}\}$ be a partition of $V(D)$ such that for each i, with $1 \leq i \leq \lfloor \frac{n}{k} \rfloor$, $V_i = \{v_j : (i-1)k \leq j \leq ik-1\}$ and, if $\lceil \frac{n}{k} \rceil \neq \lfloor \frac{n}{k} \rfloor$, $V_{\lceil \frac{n}{k} \rceil} = \{ v_j : k\lfloor \frac{n}{k} \rfloor \leq j \leq n-1\}$.

Claim 1 For every pair $v_i, v_j \in V(D)$ there is a $v_i v_j$-geodesic path T such that for every $V_p \in P$, $|V_p \cap V(T \setminus v_j)| \leq 1$.

Let $v_{rk+i}, v_{sk+j} \in V(D)$. If $r \neq s$ let $0 \leq q \leq k-1$ and t be the minimum integer such that $(r+t)k + i + q \overset{n}{\equiv} sk + j$ and let

$$T = (v_{rk+i}, v_{(r+1)k+i}, \ldots, v_{(r+t)k+i}, v_{(r+t)k+i+q})$$

be a $v_{rk+i}v_{sk+j}$-path. Since t is minimum and $0 \leq q \leq k-1$ it follows that T is a $v_{rk+i}v_{sk+j}$-geodesic path and, since for every $V_p \in P$, $|V_p| \leq k$, hence for every $V_p \in P$, $|V_p \cap V(T \setminus v_{sk+j})| \leq 1$.

If $r = s$ and $i \leq j$ it follows that $v_{rk+i}v_{sk+j} \in A(D)$ and $T = (v_{rk+i}, v_{sk+j})$ is a $v_{rk+i}v_{sk+j}$-geodesic path with the desired properties. So, let us suppose $i \geq j+1$. Thus

$$d(v_{rk+i}, v_{sk+j}) = \lceil \frac{n - k(r-s) - (i-j)}{k} \rceil = \lceil \frac{n-(i-j)}{k} \rceil.$$

Let t be the maximum integer such that $(r+t)k + i \leq n-1$. If $v_{(r+t)k+i}v_j \in A(D)$, then

$$T = (v_{rk+i}, v_{(r+1)k+i}, \ldots, v_{(r+t)k+i}, v_j, v_{k+j}, \ldots v_{sk+j})$$

is a $v_{rk+i}v_{sk+j}$-geodesic path such that for every $V_p \in P$, $|V_p \cap V(T \setminus v_{sk+j})| \leq 1$. If $v_{(r+t)k+i}v_j \notin A(D)$, since $i \geq j+1$ and t is maximum, it follows that $v_{(r+t)k+i} \in V_{\lceil \frac{n}{k} \rceil - 1}$ and $v_{(r+t)k+i}v_{n-1} \in A(D)$. Therefore

$$T = (v_{rk+i}, \ldots, v_{(r+t)k+i}, v_{n-1}, v_j, v_{k+j}, \ldots v_{sk+j})$$

is a $v_{rk+i}v_{sk+j}$-geodesic path such that for every $V_p \in P$, $|V_p \cap V(T \setminus v_{sk+j})| \leq 1$, and the claim follows.

Let $\rho : A(D) \to \{1, 2, \ldots, \lceil \frac{n}{k} \rceil\}$ be the arc-coloring of D defined as follows: for every $v_iv_j \in A(D)$, $\rho(v_iv_j) = p$ if and only if $i \in V_p$. Given $v_i, v_j \in V(D)$, from Claim 1 we see there is a v_iv_j-geodesic path T such that for every $V_i \in P$, $|V_i \cap V(T \setminus v_j)| \leq 1$ which, by definition of ρ, is a rainbow path. From here it follows that ρ is a strong rainbow coloring of D. Thus, $src^*(D) \leq \lceil \frac{n}{k} \rceil$, and since $\text{diam}(D) = \lceil \frac{n-1}{k} \rceil$, for every n such that $\lceil \frac{n}{k} \rceil = \lceil \frac{n-1}{k} \rceil$ we have $rc^*(D) = src^*(D) = \lceil \frac{n}{k} \rceil$. Hence, to end the proof just remain to verify the case $n = kt + 1$. Let suppose there is a t-rainbow coloring ρ of D, and consider $C_{(k)}$, the spanning subdigraph of D induced by the k-jumps. Since $(k, n = kt+1) = 1$ it follows that $C_{(k)}$ is a cycle, and each v_iv_{i+tk}-path in $C_{(k)}$ is the only v_iv_{i+tk}-path of length t in D. Thus, since ρ is a t-rainbow coloring, in every v_iv_{i+tk}-path in $C_{(k)}$ most appear the t colors. Therefore, for every $0 \leq i \leq n-1$, $\rho(v_iv_{i+k}) = \rho(v_{i+kt}v_{i+k(t+1)})$, which, since $(k, n = kt+1) = 1$, implies that every arc in $C_{(k)}$ receives the same color which is a contradiction. Therefore $rc^*(D) \geq t+1 = \lceil \frac{n}{k} \rceil$ and since $src^*(D) \leq \lceil \frac{n}{k} \rceil$, the theorem follows. $\square$

Now, we turn our attention on the circulant digraphs with a pair of generators $\{1, k\}$, with $2 \leq k \leq n-1$. Observe that for every circulant digraph $C_n(\{a_1, a_2\})$, if $(a_1, n) = 1$ and $b \in \mathbb{Z}_n$ is the solution of $a_1 x \overset{n}{\equiv} 1$, then $C_n(\{1, ba_2\})) \cong C_n(\{a_1, a_2\}))$. From here, we obtain the following.

Corollary 3.2. *For $k \geq 1$, $rc^*(C_{2k+1}(1, k+1)) = src^*(C_{2k+1}(1, k+1)) = k+1$.*

Proof. By Theorem 3.1, for every $n \geq 4$, $rc^*(C_n([2]) = src^*(C_n([2])) = \lceil \frac{n}{2} \rceil$. Since $(k+1, 2k+1) = 1$ and 2 is the solution of $(k+1)x \overset{2k+1}{\equiv} 1$, then $C_{2k+1}(\{1, k+1\})) \cong C_{2k+1}(\{1, 2\})) = C_{2k+1}([2])$ and the result follows. $\square$

Observe that given any circulant digraph $C_n(\{1, k\})$, for every pair $v_i, v_j \in C_n(\{1, k\})$ we have $d(v_i, v_j) = d(v_0, v_{j-i})$ (where $j - i$ is taken modulo n). Thus, $\text{diam}(C_n(\{1, k\})) = max\{d(v_0, v_i) : v_i \in V(C_n(\{1, k\}))\}$.

Given two positive integers i, k, let denote as $re(i, k)$ the residue of i modulo k.

Lemma 3.3. *Let $C_n(\{1,k\})$ be a circulant digraph and $V = \{v_0, \ldots, v_{n-1}\}$ it set of vertices . If $n \geq (k-1)\lceil \frac{n}{k} \rceil$ then for every $v_i \in V$, $d(v_0, v_i) = \lfloor \frac{i}{k} \rfloor + re(i,k)$.*

Moreover diam($C_n(\{1,k\})$) $= \lfloor \frac{n-1}{k} \rfloor + max\{re(n-1,k), k-2\}$.

Proof. Let $v_i \in V$, $P = (v_0 = u_0, u_1 \ldots, u_s = v_i)$ be a v_0v_i-geodesic path with a minimum number of k-jumps, and suppose in P there are p k-jumps and q 1-jumps. Also suppose the first p steps of P are k-jumps, and the last q are 1-jumps. Thus $d(v_0, v_i) = p + q$. Since P is geodesic, it follows that $q \leq k-1$ and therefore $p \geq \lfloor \frac{i}{k} \rfloor$. Hence $v_{k\lfloor \frac{i}{k} \rfloor} = u_{\lfloor \frac{i}{k} \rfloor} \in V(P)$ and the subpath

$$Q = (v_{k\lfloor \frac{i}{k} \rfloor} = u_{\lfloor \frac{i}{k} \rfloor} \ldots, u_j, \ldots, u_s = v_i)$$

is a $v_{k\lfloor \frac{i}{k} \rfloor}v_i$-geodesic path with $p' = p - \lfloor \frac{i}{k} \rfloor$ k-jumps and $d(v_{k\lfloor \frac{i}{k} \rfloor}, v_i) = p' + q \leq i - k\lfloor \frac{i}{k} \rfloor = re(i,k)$. If $p > \lfloor \frac{i}{k} \rfloor$ then $q < re(i,k)$ and since $re(i,k) < k$, it follows that $p' \geq \lceil \frac{n}{k} \rceil$. Therefore, if $m = k\lceil \frac{n}{k} \rceil - n$, $v_{k\lfloor \frac{i}{k} \rfloor + m} = u_{\lfloor \frac{i}{k} \rfloor + \lceil \frac{n}{k} \rceil} \in V(Q)$ and the subpath

$$(v_{k\lfloor \frac{i}{k} \rfloor} = u_{\lfloor \frac{i}{k} \rfloor} \ldots, u_j, \ldots, u_{\lfloor \frac{i}{k} \rfloor + \lceil \frac{n}{k} \rceil} = v_{k\lfloor \frac{i}{k} \rfloor + m})$$

is a $v_{k\lfloor \frac{i}{k} \rfloor}v_{k\lfloor \frac{i}{k} \rfloor + m}$-geodesic path of $\lceil \frac{n}{k} \rceil$ k-jumps and $d(v_{k\lfloor \frac{i}{k} \rfloor}, v_{k\lfloor \frac{i}{k} \rfloor + m}) = \lceil \frac{n}{k} \rceil \leq m$. Since $n \geq (k-1)\lceil \frac{n}{k} \rceil$ it follows that $\lceil \frac{n}{k} \rceil \geq k\lceil \frac{n}{k} \rceil - n = m$ and therefore $d(v_{k\lfloor \frac{i}{k} \rfloor}, v_{k\lfloor \frac{i}{k} \rfloor + m}) = m$. Thus, replacing in P the subpath

$$(v_{k\lfloor \frac{i}{k} \rfloor} = u_{\lfloor \frac{i}{k} \rfloor} \ldots, u_j, \ldots, u_{\lfloor \frac{i}{k} \rfloor + \lceil \frac{n}{k} \rceil} = v_{k\lfloor \frac{i}{k} \rfloor + m})$$

by the subpath

$$(v_{k\lfloor \frac{i}{k} \rfloor}, v_{k\lfloor \frac{i}{k} \rfloor + 1}, \ldots, v_{k\lfloor \frac{i}{k} \rfloor + m})$$

we obtain a v_0v_i-geodesic path with less k-jumps than P, which is a contradiction. Thus $p = \lfloor \frac{i}{k} \rfloor$ and therefore $q = re(i,k)$ which implies that $d(v_0, v_i) = \lfloor \frac{i}{k} \rfloor + re(i,k)$ and the first part of the result follows. For the second part, first observe that $d(v_0, v_{n-1}) = \lfloor \frac{n-1}{k} \rfloor + re(n-1,k)$ and $d(v_0, v_{(k\lfloor \frac{n-1}{k} \rfloor)-1}) = \lfloor \frac{n-1}{k} \rfloor + k - 2$, thus diam($C_n(\{1,k\})$) $\geq \lfloor \frac{n-1}{k} \rfloor + max\{re(n-1,k), k-2\}$. If there is $v_i \in V$ such that $d(v_0, v_i) > \lfloor \frac{n-1}{k} \rfloor + k - 2$, it follows that $n - 1 \geq i \geq k\lfloor \frac{n-1}{k} \rfloor$ but then $d(v_0, v_i) \leq d(v_0, v_{n-1}) = \lfloor \frac{n-1}{k} \rfloor + re(n-1,k)$ and the result follows. □

Theorem 3.4. *For every integer $k \geq 2$*

(*i*) $rc^*(C_{2k}(\{1,k\})) = src^*(C_{2k}(\{1,k\})) = k.$

(*ii*) $rc^*(C_{2k}(\{1,k+1\})) = src^*(C_{2k}(\{1,k+1\})) = k.$

Proof. Let $V = \{v_0, \ldots, v_{2k-1}\}$ be the set of vertices of $C_{2k}(\{1,k\}$. By Lemma 3.3 we see that $k = \operatorname{diam}(C_{2k}(\{1,k\}))$ and therefore $k \leq rc^*(C_{2k}(\{1,k\}))$. Let $\{V_0, \ldots, V_{k-1}\}$ be a partition of V, where $V_r = \{v_r, v_{r+k}\}$ for $0 \leq r \leq k-1$ and define a k-colouring ρ such that for every $0 \leq r \leq k-1$, $(u,u') \in \rho^{-1}(r)$ if $u \in V_r$. Let $v_i, v_j \in V$ and suppose $i+q+pk \overset{n}{\equiv} j$ where $d(v_i,v_j) = p+q$ and $q \leq k-1$. Observe that, since $q < k$, $v_i C_{(1)} v_{i+q} C_{(k)} v_{i+pk+q}$ is a rainbow v_iv_j-path and by Lemma 3.3 is v_iv_j-geodesic. Therefore $src^*(C_{2k}(\{1,k\})) \leq k$ and (i) follows. For (ii), let $V = \{v_0, \ldots, v_{2k-1}\}$ be the set of vertices of $C_{2k}(\{1,k+1\})$ and let $\{V_0, \ldots, V_{k-1}\}$ as before. By Lemma 3.3 it follows that $\operatorname{diam}(C_{2k}(\{1,k+1\})) = k$ which implies $k \leq rc^*(C_{2k}(\{1,k+1\}))$. Now let ρ be a k-colouring such that $(u,u') \in \rho^{-1}(r)$ if $u \in V_r$. Since $N^+(u) = V_{r+1}$ for each $u \in V_r$ (taken $r+1$ modulo k), it follows that every path of length at most k is rainbow, in particular every geodesic path is rainbow. Thus $k \geq src^*(C_{2k}(\{1,k+1\}))$ and (ii) follows. □

Theorem 3.5. *For every integer $k \geq 3$ we have*

$$src^*(C_{(k-1)^2}(\{1,k\})) = rc^*(C_{(k-1)^2}(\{1,k\})) = 2k-4.$$

Proof. By Lemma 3.3 we see that $\operatorname{diam}(C_{(k-1)^2}(\{1,k\})) = 2k-4$ and therefore $rc^*(C_{(k-1)^2}(\{1,k\})) \geq 2k-4$. Let $V = \{v_0, \ldots, v_{(k-1)^2-1}\}$ be the set of vertices of $C_{(k-1)^2}(\{1,k\})$ and for each i, with $0 \leq i < (k-1)^2$, identify the vertex v_i with the pair $\langle \lfloor \frac{i}{k-1} \rfloor, re(i,k-1) \rangle$. Let $\mathcal{V} = \{V_0, \ldots, V_{k-2}\}$ be a partition of V, where $V_r = \{\langle r,s \rangle \mid 0 \leq s \leq k-2\}$ for $0 \leq r \leq k-2$, and let ρ be a $(2k-4)$-colouring defined as follows: For each r with $0 \leq r \leq k-1$,

1. The arc $(\langle r,s \rangle \langle r+1,s \rangle)$, with $0 \leq s \leq k-2$, receives color r.

2. The arcs $(\langle r,0 \rangle \langle r,1 \rangle)$ and $(\langle r,k-2 \rangle \langle r+1,0 \rangle)$ receive colour r.

3. The arc $(\langle r,s \rangle \langle r,s+1 \rangle)$, with $1 \leq s \leq k-3$, receives colour $k-2+s$.

Observe that every path with length at most $k-1$ in $C_{(k)}$ is rainbow, and, except for those paths of lenght $k-1$ in $C_{(1)}$ starting at $\langle r, 0\rangle$ (with $0 \leq r < k-1$), every path in $C_{(1)}$ with lenght at most $k-1$ is rainbow. From the structure of ρ we see that to prove ρ is a strong coloring we just need to show that for every $v \in V_0$ and every $w \in V$ there is a rainbow vw-geodesic path.

Let $\langle 0, s_0\rangle \in V_0$ and $\langle r, s\rangle \in V_r$. Since $\langle 0, s_0\rangle = v_{s_0}$ and $\langle r, s\rangle = v_{r(k-1)+s}$, by Lemma 3.3,

$$d(v_{s_0}, v_{r(k-1)+s}) = \lfloor \frac{(k-1)r + s - s_0}{k} \rfloor + re((k-1)r + s - s_0, k)$$

(taken $(k-1)r + s - s_0$ modulo $(k-1)^2$). Thus, if $t = \lfloor \frac{(k-1)r+s-s_0}{k} \rfloor$,

$$P = \langle 0, s_0\rangle C_{(k)} \langle \lfloor \frac{s_0 + tk}{k-1} \rfloor, re(s_0 + tk, k-1)\rangle C_{(1)} \langle r, s\rangle$$

is a geodesic path. The subpath in $C_{(k)}$ receives colors j, with $0 \leq j \leq \lfloor \frac{s_0+tk}{k-1} \rfloor - 1 \leq k-2$, and the subpath in $C_{(1)}$ receives colors i. with $k-1 \leq i \leq 2k-3$ or $i = \lfloor \frac{s_0+tk}{k-1} \rfloor$. Thus, if P is not rainbow then we have that: the subpath in $C_{(1)}$ most be of lenght $k-1$; $\langle \lfloor \frac{s_0+tk}{k-1} \rfloor, re(s_0 + tk, k-1)\rangle = \langle r-1, 0\rangle$ and $\langle r, s\rangle = \langle r, 0\rangle$.

If $r-1 = 0$ it follows that $\langle 0, s_0\rangle = \langle 0, 0\rangle$ and the path Q of k-jumps $\langle 0, 0\rangle C_{(k)} \langle 1, 0\rangle$ of lenght $k-1$ is a geodesic rainbow. If $r-1 = 1$, $(\langle 0, s_0\rangle \langle 1, 0\rangle)$ most be a k-jump which is not possible. If $r-1 \geq 2$, let Q be the rainbow geodesic obtained by the concatenation of the paths $\langle 0, s_0\rangle C_{(k)} \langle r-3, k-2\rangle$ (which receives colors between 0 and $r-4$); the arcs $(\langle r-3, k-2\rangle, \langle r-2, 0\rangle)$ and $(\langle r-2, 0\rangle, \langle r-1, 1\rangle)$ (with colors $r-3$ and $r-2$ respectively); and $\langle r-1, 1\rangle C_{(1)} \langle r, 0\rangle$ (which receives the colors $r-1$ and $k-1, \ldots, 2k-3$). Hence, P or Q is a $\langle 0, s_0\rangle\langle r, s\rangle$-geodesic rainbow, and the theorem follows. □

Theorem 3.6. *If $n = a_n k$ with $a_n \geq k-1 \geq 2$, then*

$$src^*(C_n(\{1, k\})) = rc^*(C_n(\{1, k\})) = a_n + k - 2.$$

Proof. By Lemma 3.3 we see that $\text{diam}(C_n(\{1, k\})) = a_n + k - 2$ and then to prove the result just remain to show that $src^*(C_n(\{1, k\})) \leq a_n + k - 2$. Let $V = \{v_0, \ldots, v_{n-1}\}$

be the set of vertices of $C_n(\{1,k\})$ and, for each i, with $0 \le i < n$, identify the vertex v_i with the pair $\langle \lfloor \frac{i}{k} \rfloor, re(i,k)\rangle$. Let $\{V_0, \dots, V_{a_n-1}\}$ be a partition of V, where $V_r = \{\langle r,s\rangle : 0 \le s < k\}$ for $0 \le r < a_n$, and let ρ be a (a_n+k-2)-colouring defined as follows: For each r, with $0 \le r \le a_n - 1$, let

1. The arc $(\langle r,s\rangle\langle r+1,s\rangle)$, with $0 \le s < k$, receives color r.
2. If $r \ge k-2$ the arcs $(\langle r,0\rangle\langle r,1\rangle)$ and $(\langle r,k-1\rangle\langle r+1,0\rangle)$ receive color r; and, for each $1 \le j \le k-2$, the arc $(\langle r,j\rangle\langle r,j+1\rangle)$ receives color $a_n - 1 + j$.
3. If $r \le k-3$ the arc $(\langle r,k-2-r\rangle\langle r,k-1-r\rangle)$ receives color r; for each $0 \le j \le k-3-r$ the arc $(\langle r,j\rangle\langle r,j+1\rangle)$ receives color $a_n + r + j$; for each $k-1-r \le j \le k-2$ the arc $(\langle r,j\rangle\langle r,j+1\rangle)$ receives color $a_n + j - (k-1-r)$; and the arc $(\langle r,k-1\rangle\langle r+1,0\rangle)$ receives color $a_n + r$.

Observe that for every pair $1 \le r, r' < a_n$ the path $\langle r,s\rangle C_{(k)}\langle r',s\rangle$ is a rainbow path with colors $r, r+1, \dots, r'-1$ (taken the sequence modulo a_n). Also every path P of lenght at most $k-1$ in $C_{(1)}$ is rainbow. Moreover, if for some $0 \le r < a_n$, $V(P) \subseteq V_r$ then the colors appearing in P are contained in $\{a_n, \dots, a_n+(k-3)\} \cup \{r\}$; and if $V(P)$ starts at V_r and ends at V_{r+1}, the colors of P are in $\{a_n, \dots, a_n+(k-3)\} \cup \{r, r+1\}$.

Let $\langle r,s\rangle$ and $\langle r',s'\rangle$ be distinct vertices of $C_n(\{1,k\})$. If $r \ne r'$ it is not hard to see that either $\langle r,s\rangle C_{(k)}\langle r',s\rangle C_{(1)}\langle r',s'\rangle$ (if $s \le s'$) or $\langle r,s\rangle C_{(k)}\langle r'-1,s\rangle C_{(1)}\langle r',s'\rangle$ (if $s > s'$) is a rainbow $(\langle r,s\rangle\langle r',s'\rangle)$-path. If $r = r'$ and $s < s'$ we see that $\langle r,s\rangle C_{(1)}\langle r,s'\rangle$ is a rainbow path. Let us suppose $r = r'$ and $s > s'$. If no arc $(\langle r,t\rangle\langle r,t+1\rangle)$, with $0 \le t < s'$, receives color r, the path $\langle r-1,s\rangle C_{(1)}\langle r,s'\rangle$ receive only colors in $\{a_n, \dots, a_n+(k-3)\} \cup \{r-1\}$, and therefore $\langle r,s\rangle C_{(k)}\langle r-1,s\rangle C_{(1)}\langle r,s'\rangle$ is a rainbow path. If some arc $(\langle r,t\rangle\langle r,t+1\rangle)$, with $0 \le t < s'$, receives color r, by definition of ρ most be either $(\langle r,0\rangle\langle r,1\rangle)$ (if $r \ge k-2$), or $(\langle r,k-2-r\rangle\langle r,k-1-r\rangle)$ (if $r \le k-3$). For the first case in the path $P = \langle r,s\rangle C_{(k)}\langle a_n-1,s\rangle C_{(1)}\langle 0,s'\rangle C_{(k)}\langle r,s'\rangle$, the k-jumps receive colors $\{0, \dots, r, \dots, a_n-2\}$ and, by definition of ρ, the only 1-jump of color 0 is $(\langle 0,k-2\rangle\langle 0,k-1\rangle)$. Thus, since $s' < s \le k-1$, the colors appearing in $\langle a_n-1,s\rangle C_{(1)}\langle 0,s'\rangle$ are contain in $\{a_n, \dots, a_n+(k-3)\} \cup \{a_n-1\}$ and therefore P is rainbow. For the second case in the path $P = \langle r,s\rangle C_{(k)}\langle k-2-s,s\rangle C_{(1)}\langle k-1-$

$s, s'\rangle C_{(k)}\langle r, s'\rangle$ the k-jumps receive colors $\{0, \ldots, k-3-s, k-1-s, \ldots, a_n - 1\}$ and, since $s > s' > t \geq 0$, $k-1-s \leq k-3$ and therefore the only 1-jump of color $k-1-s$ is $(\langle k-1-s, s-1\rangle\langle k-2-s, s\rangle)$. Thus the colors in $\langle k-2-s, s\rangle C_{(1)}\langle k-1-s, s'\rangle$ are contain in $\{a_n, \ldots, a_n + (k-3)\} \cup \{k-2-s\}$ and P is a rainbow path. In all the cases, from Lemma 3.3 we see that all the paths are geodesic and the result follows. □

References

[1] J. Bang-Jensen, G. Gutin. *Digraphs*. Springer, 2009.

[2] S. Chakraborty, E. Fischer, A. Matsliah, R. Yuster. Hardness and algorithms for rainbow connection. *J. Comb. Optim.* **21 (3)** (2011), 330– 347.

[3] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang. Rainbow connection in graphs, *Math. Bohemica.* **133(1)**(2008), 85–98.

[4] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang. The rainbow connectivity of a graph, *Networks* **54**(2009), 75–81.

[5] P. Dorbec, I. Schiermeyer, E. Sidorowicz, E. Sopena. Rainbow connection in oriented graphs, *Discrete Applied Math.*. In press

[6] I. Schiermeyer. Bounds for the rainbow connection number of graphs, *Discuss. Math. Graph Theory* **30**(2011). 387–395.

[7] I. Schiermeyer. Rainbow connection and minimum degree, *Discrete Applied Math.* **161 (12)** (2013). 387–395.

[8] A, Kemnitz, I. Schiermeyer. Graphs with rainbow connection number two, *Discuss Math. Graph Theory* **63** (2010), 185–191.

[9] M. Krivelevich, R. Yuster. The rainbow connection of a graph is (at most) reciprocal to its minimum degree, *J. Graph Theory* 63 (2009), 185–191.

[10] X. Li, M. Liu, I. Schiermeyer. Rainbow connection number of dense graphs. *Discuss. Math. Graph Theory* **33 (3)** (2013), 603–611.

[11] X. Li, S. Liu. A sharp upper bound for the rainbow 2-connection number of a 2-connected graph. *Discrete Math.* **313 (6)** (2013), 755–759.

[12] X. Li, Y. Shi. Rainbow connection in 3-connected graphs, *Graphs Combin* **29 (5)** (2013), 1471–1475.

[13] X. Li, Y. Sun. Rainbow connections of graphs - A survey, *Graphs Combin* **29 (1)** (2013), 1–38.